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NOTE

MINOR CHARACTERIZATION OF UNDIRECTED BRANCHING GREEDOIDS – A SHORT PROOF*

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In 1959 Tutte [5] gave a minor characterization of graphic matroids. Within the framework of greedoids, a natural analogue of the cycle matroid in graphs is the branching greedoid. Schmidt has shown that, similar to Tutte's result, branching greedoids can be characterized by forbidden minors. Here we give a simpler proof of this theorem.

1. Introduction

In 1959 Tutte [5] gave a minor characterization of graphic matroids. As a generalization of matroids, Korte and Lovász [1–3] introduced the concept of greedoids. In this framework, a natural analogue of the cycle matroid of a graph (where the independent sets are forests) is the branching greedoid with rooted trees as feasible sets. Schmidt [4] has shown that, similar to Tutte's result, branching greedoids can be characterized by forbidden minors. The purpose of this paper is to give a simpler proof of Schmidt's theorem.

Throughout this paper, let E be a finite set and $\mathcal{F} \subseteq 2^E$ a set system with $\emptyset \in \mathcal{F}$ and for any two sets $X, Y \in \mathcal{F}$ with $|X| < |Y|$ there exists a $y \in Y \backslash X$ such that $X \cup y \in \mathcal{F}$. $(E, \mathcal{F})$ is called a *greedoid*. The members of $\mathcal{F}$ are called *feasible sets*. For $X \in \mathcal{F}$ there exists a *feasible ordering* $x_1 \cdots x_k$, $k = |X|$, such that $\{x_1, \ldots, x_i\} \in \mathcal{F}$ for $1 \leqslant i \leqslant k$. We use greek letters to denote finite string (*words*) of elements (*letters*) of E and write $\alpha \in \mathcal{F}$ if α is a feasible ordering of a member of $\mathcal{F}$. Maximal feasible sets (words) are called *basic*, letters not occurring in any feasible word are *loops*.

We call $(E, \mathcal{F})$ an *interval greedoid* if for all $A, B, C \in \mathcal{F}$ with $A, B \subseteq C$ also $A \cup B \in \mathcal{F}$. If in addition $A \cap B \in \mathcal{F}$, then $(E, \mathcal{F})$ is called a *local poset greedoid*.

For $A \subseteq E$ let $\mathcal{F}\backslash A := \{X \in \mathcal{F}, X \subseteq E\backslash A\}$. Then the *restriction* $(E\backslash A, \mathcal{F}\backslash A)$ is a greedoid. For $A \in F$ let $F/A := \{X \subseteq E\backslash A\colon A \cup X \in F\}$, then $(E\backslash A, F/A)$ is a greedoid and is called the *contraction* of A in $(E, \mathcal{F})$. $(E', \mathcal{F}')$ is a *minor* of $(E, \mathcal{F})$ if there exist $F \in \mathcal{F}$, $A \subseteq E\backslash F$ such that $E' = E\backslash(A \cup F)$ and $\mathcal{F}' = (\mathcal{F}/F)\backslash A$.

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Lemma 1. *An interval greedoid is a local poset greedoid if and only if it does not contain the following minor*

$$(A) \qquad E' = \{x, y, z\}, \qquad \mathcal{F}' = 2^E \setminus \{z\}$$

Proof. Korte and Lovász [3]. □

A feasible set $X \in \mathcal{F}$ is called a *path* if for a unique $a \in X$ we have $X \setminus a \in \mathcal{F}$. We then say that X is an *a-path*. The collection of paths is denoted by $\mathcal{P}$ and $\mathcal{P}/\alpha$ are the paths in the contraction $\mathcal{F}/\alpha$.

Lemma 2. *An interval greedoid $(E, \mathcal{F})$ is a local poset greedoid if and only if every $X \in \mathcal{F}$ contains a unique a-path for every $a \in X$.*

Proof. Clearly, local poset greedoids have this property. Conversely, consider feasible sets $A, B, C \in \mathcal{F}$ with $A, B \subseteq C$. Then $A \cup B \in \mathcal{F}$ contains a unique a-path Z_a for every $a \in A \cap B$. Hence $Z_a \subseteq A \cap B$ and

$$A \cap B = \bigcup \{Z_a : a \in A \cap B\} \in \mathcal{F}. \quad \square$$

Lemma 3. *For a local poset greedoid $(E, \mathcal{F})$ the following statements are equivalent:*

(i) *Every path has a unique feasible ordering,*

(ii) *$(E, \mathcal{F})$ does not contain the following minor*

$$(B) \qquad E = \{x, y, z\}, \qquad \mathcal{F} = \{\emptyset, x, y, \{x, y\}, \{x, y, z\}\}.$$

Proof. (i) $\Rightarrow$ (ii) obvious.

(ii) $\Rightarrow$ (i) It suffices to show that for every z-path X, $X \setminus z$ is a path. Suppose X is a z-path and $(X \setminus z) \setminus x \in \mathcal{F}$ and $(X \setminus z) \setminus y \in \mathcal{F}$ for some $x, y \in X \setminus z$. Then, by the local poset property, $A = X \setminus \{x, y, z\} \in \mathcal{F}$. But then (B) is a minor in $\mathcal{F}/A$. □

We say two elements $x, y \in E$ are *parallel* ($x \parallel y$) in $\mathcal{F}$ if $\alpha x \beta \in \mathcal{F}$ if and only if $\alpha y \beta \in \mathcal{F}$ for all strings α, β.

Let $G = (V, E)$ be a connected graph with a specified *root* $r \in V$ and let $\mathcal{F} \subseteq 2^E$ be the collection of trees incident with r. Then $(E, \mathcal{F})$ is a local poset greedoid, the paths of $\mathcal{F}$ are precisely the (node disjoint) paths in G with endnode r. $(E, \mathcal{F})$ is the *branching greedoid* of G with root $r \in V$.

For more examples and a detailed exposition of structural properties of greedoids, see Korte and Lovász [1–3].

2. The main theorem

Theorem. *An interval greedoid is a branching greedoid if and only if it does not contain one of the following minors*

(*A*) $\mathscr{F} = 2^{\{x,y,z\}} \setminus \{z\}$,

(*B*) $\mathscr{F} = \{\emptyset, x, y, \{x, y\}, \{x, y, z\}\}$,

(*C*) $\mathscr{F} = \{\emptyset, x, y, \{x, y\}, \{x, z\}\}$,

(*D*) $\mathscr{F} = 2^{\{x,y,z\}} \setminus \{x, y, z\}$.

It is straightforward to check that none of the above forbidden minors are branching greedoids. Hence the condition is necessary. For the proof of sufficiency we need some preparatory lemmas.

Lemma 4. *Let* $(E, \mathscr{F})$ *be a local poset greedoid and let* $x \in \mathscr{F}$, $x \notin \alpha$.

(*a*) *If* $\alpha \in \mathscr{P}$ *and* $\alpha \in \mathscr{F}/x$ *then* $\alpha \in \mathscr{P}/x$.

(*b*) *If* $\alpha \in \mathscr{P}/x$ *then either* $\alpha \in \mathscr{P}$ *or* $x\alpha \in \mathscr{P}$.

Proof. Let u be the last letter of α.

(a) Choose a u-path $\alpha' u$ in α with respect to $\mathscr{F}/x$. Denote by A and A' the sets underlying α and α', resp. Then $A, A' \cup \{u, x\} \subseteq A \cup x$ and the local intersection property implies $\alpha' u \in \mathscr{F}$. Hence, by Lemma 2, $A' \cup u = A$ and $\alpha \in \mathscr{P}/x$.

(b) Choose a u-path $\alpha' u$ in $x\alpha$ with respect to $\mathscr{F}$. If $x \in \alpha'$, then $A' \cup u \setminus x$ is a path in $\mathscr{F}/x$, thus $A' \cup u \setminus x = A$ and $x\alpha \in \mathscr{P}$. If $x \notin \alpha' u$, then $x\alpha' u \in \mathscr{F}$ since $x, A' \cup u \subseteq A \cup x \in \mathscr{F}$. This implies that $\alpha' u \in \mathscr{F}/x$ and $A' \cup u = A$ and $\alpha \in \mathscr{P}/x$. $\square$

Lemma 5. *Let* $(E, \mathscr{F})$ *be an interval greedoid without minors* (*A*)–(*C*). *Suppose that* $\alpha = x_1 \cdots x_l \in \mathscr{P}$, $x_0 \in \mathscr{F}$ *and* $\{x_0, x_1, \ldots, x_{k-1}\} \in \mathscr{F}$, $\{x_0, x_1, \ldots, x_k\} \notin \mathscr{F}$ *for some* $k \in \{1, \ldots, l\}$. *Then* $x_0 x_k x_{k-1} \cdots x_2 \in \mathscr{P}$, $x_0 x_{k+1} \cdots x_l \in \mathscr{P}$ *and*

$$\{x_0, x_1, \ldots, x_l\} \setminus \{x_i, \ldots, x_j\} \in \mathscr{F} \quad \textit{for } 1 \leqslant i \leqslant j \leqslant k.$$

Proof. For $k = 1$ it suffices to prove that $x_0 x_2 \cdots x_l \in \mathscr{P}$. To see this, augment $x_0 \in \mathscr{F}$ successively from $x_1 \cdots x_l \in \mathscr{F}$. Since $\{x_0, x_1\} \notin \mathscr{F}$ and by the interval property, $x_0 x_2 \cdots x_l \in \mathscr{P}$.

For $k \geqslant 2$ consider $\mathscr{F}/x_1$. Then $x_2 \cdots x_l$, x_0 satisfy the requirement of our lemma, hence, by induction we get

$$x_0 x_k x_{k-1} \cdots x_3 \in \mathscr{P}/x_1, \tag{2.1}$$

$$x_0 x_{k+1} \cdots x_l \in \mathscr{P}/x_1, \tag{2.2}$$

$$\{x_0, x_2, \ldots, x_l\} \setminus \{x_i, \ldots, x_j\} \in \mathscr{F}/x_1 \quad \text{for } 2 \leqslant i \leqslant j \leqslant k. \tag{2.3}$$

Since $\{x_0, x_1, \ldots, x_{k-1}\} \in \mathscr{F}$ and $x_1 \cdots x_{k-1} \in \mathscr{P}$, Lemma 4 implies

$$x_1 \cdots x_{k-1} \in \mathscr{P}/x_0. \tag{2.4}$$

Now

$$x_k \in \mathscr{P}/x_0. \tag{2.5}$$

For $k \geqslant 3$, this immediately follows from (2.1). For $k = 2$, (2.5) also holds, since otherwise $\{x_0, x_1, x_2\}$ induces a minor of type (C) in $\mathscr{F}$.

From (2.3) we deduce that

$$\{x_k, x_1, \ldots, x_{k-2}\} \in \mathscr{F}/x_0 \quad \text{and } \{x_k, x_1, \ldots, x_{k-1}\} \notin \mathscr{F}/x_0. \tag{2.6}$$

Hence the induction hypothesis implies $x_k x_{k-1} \cdots x_2 \in \mathscr{P}/x_0$ and

$$\{x_k x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_l\} \setminus \{x_1, \ldots, x_j\} \in \mathscr{F}/x_0 \quad \text{for } 1 \leqslant j \leqslant k-1. \tag{2.7}$$

Thus, using Lemma 4, $x_0 x_k x_{k-1} \cdots x_2 \in \mathscr{P}$. Moreover, (2.2) and Lemma 4 yield $x_0 x_{k+1} \cdots x_l \in \mathscr{P}$ and finally (2.3) and (2.7) imply $\{x_0, x_1, \ldots, x_l\} \setminus \{x_i, \ldots, x_j\} \in \mathscr{F}$ for $1 \leqslant i \leqslant j \leqslant k$. $\square$

As an immediate consequence we obtain

Corollary 6. *Let $(E, \mathscr{F})$ be as in Lemma 5 and let $\alpha\beta x, \alpha\gamma y \in \mathscr{P}$ with $\gamma \neq \emptyset$. Then $\alpha\beta\gamma x, \alpha\beta\gamma y \in \mathscr{F}$ and $\alpha\beta x y \notin \mathscr{F}$ imply $\alpha\beta x y \in \mathscr{P}$.*

Proof. By Lemma 4, $\gamma y \in \mathscr{P}/\alpha\beta$ and furthermore $x, \gamma x \in \mathscr{F}/\alpha\beta$, $\gamma y x \neq \mathscr{F}/\alpha\beta$. Then Lemma 5 implies $xy \in \mathscr{P}/\alpha\beta$. Hence, again by Lemma 4, $\alpha\beta x y \in \mathscr{P}$. $\square$

Lemma 7. *Let $(E, \mathscr{F})$ be an interval greedoid without minors (A)–(D). Then $x \parallel y$ in $\mathscr{F}$ if and only if either x and y are loops or there exist paths αx, αy such that $\alpha x y \notin \mathscr{F}$.*

Proof. If $x \parallel y$ in $\mathscr{F}$ and x, y are not loops, then for any x-path αx we have $\alpha y \in \mathscr{P}$ and $\alpha x y \notin \mathscr{F}$.

Suppose now that αx, $\alpha y \in \mathscr{P}$ and $\alpha x y \notin \mathscr{F}$. If $\alpha = \emptyset$, then the interval property immediately implies $x \parallel y$ in $\mathscr{F}$.

Now let $\alpha = a\alpha'$ and suppose that x, y are not parallel in $\mathscr{F}$. Choose minimal words β, γ such that $\beta x \gamma \in \mathscr{F}$, $\beta y \gamma \notin \mathscr{F}$. Then $\beta x \in \mathscr{P}$, for if $\beta' x \in \mathscr{P}$ with $\beta = \beta'\beta''$, then $\beta' x \beta'' \gamma \in \mathscr{F}$. Hence, by the minimality of β, $\beta' y \beta'' \gamma \in \mathscr{F}$ and, by the interval property, $\beta y \gamma \in \mathscr{F}$. Clearly $\beta \neq \emptyset$. Furthermore, we claim that we may assume $\gamma = \emptyset$ or $\gamma = y$. If $\gamma = \gamma' b$, then by minimality of γ, $\beta y \gamma' \in \mathscr{F}$ and, since $\beta y \gamma \notin \mathscr{F}$, $\beta y \gamma' x \in \mathscr{F}$. By the interval property, this implies that $\beta x y \in \mathscr{F}$. Since $\beta y y \notin \mathscr{F}$, we thus may assume $\gamma = y$.

We now distinguish two cases.

(i) $\beta x \in \mathscr{P}/a$ or $\beta y \in \mathscr{P}/a$. Then both $\beta x \in \mathscr{P}/a$ and $\beta y \in \mathscr{P}/a$ since $x \parallel y$ in $\mathscr{F}/a$ by induction. By Lemma 4, $\beta x \in \mathscr{P}$ and $\beta y \in \mathscr{P}$ and $\beta y \in \mathscr{P}$, in particular $\gamma = y$. But then $\{a, x, y\}$ induce minor (D) in $\mathscr{F}/\beta$.

(ii) $\beta x, \beta y \notin \mathscr{P}/a$. From Lemma 4 we get $a\beta x, a\beta y \notin \mathscr{F}$. Let B be the set underlying β.

For $\gamma = y$, augment $a \in \mathscr{F}$ from $\beta xy \in \mathscr{F}$. Then necessarily $B \cup \{a, x, y\} \backslash b \in \mathscr{F}$ for some $b \in B$, contradicting $x \parallel y$ in $\mathscr{F}/a$. Thus $\gamma = \emptyset$. Let β' be the longest initial substring of β such that $\beta' a \in \mathscr{F}$. If $\beta' = \beta$, apply Lemma 5 to $\beta x \in \mathscr{P}$, $a \in \mathscr{F}$, $\beta xa \notin \mathscr{F}$. Then $ax\delta \in \mathscr{P}$, where δ is the reverse order of β without the first letter b of β. Since $x \parallel y$ in $\mathscr{F}/a$, also $ay\delta \in \mathscr{P}$. A second application of Lemma 5 to $ay\delta \in \mathscr{P}$, $b \in \mathscr{F}$, $ay\delta b \notin \mathscr{F}$ yields $\beta y \in \mathscr{P}$, contradicting $\gamma = \emptyset$.

Suppose now that $\beta = \beta' b \beta''$. In $\mathscr{F}/\beta'$, a and b are parallel. Hence $\beta' b \beta'' x \in \mathscr{F}$ implies $\beta' a \beta'' x \in \mathscr{F}$. Since $x \parallel y$ in $\mathscr{F}/a$, we get $\beta' a \beta'' y \in \mathscr{F}$. Again, by $a \parallel b$ in $\mathscr{F}/\beta'$, we finally conclude $\beta' b \beta'' y = \beta y \in \mathscr{F}$, contradiction. □

Lemma 8. *Let $(E, \mathscr{F})$, $(E, \mathscr{F}')$ be interval greedoids without minors (A)–(D). Suppose that $\mathscr{P} = \mathscr{P}'$ and that $x \parallel y$ in $\mathscr{F}$ if and only if $x \parallel y$ in $\mathscr{F}'$, then $\mathscr{F} = \mathscr{F}'$.*

Proof. Suppose $\mathscr{F} \neq \mathscr{F}'$. Choose a word $\alpha x \in \mathscr{F} \backslash \mathscr{F}'$ of minimal length. We may assume that $\alpha = \alpha_1 \alpha_2 y$ with $\alpha_1 x \in \mathscr{P}$. Let $\beta_1 \beta_2 y$ be the y-path in α. Then β_1 must be an initial substring of α_1 and by the interval property we may assume that β_2 is an initial substring of α_2. Set $\alpha_1 = \beta_1 \gamma$ and $\alpha_2 = \beta_2 \delta$.

If $\delta = \emptyset$, then we can apply Corollary 6 to $\beta_1 \gamma x$, $\beta_1 \beta_2 y \in \mathscr{P}'$, $\alpha_1 \beta_2 x$, $\alpha_1 \beta_2 y \in \mathscr{F}'$, $\beta_1 \gamma \beta_2 yx \notin \mathscr{F}'$. Hence either $\beta_1 \gamma xy \in \mathscr{P}'$ or $\beta_1 \beta_2 yx \in \mathscr{P}'$ since β_2 and γ cannot both be empty. But then $\alpha x \in \mathscr{F}$ contains either two x-paths or two y-paths, contradicting Lemma 2.

Now let $\delta = \delta' z$. Using the minimality of α and the interval property, we observe that $\{x, y, z\}$ induces a minor of type (D) in $\mathscr{F}/\alpha_1 \gamma \delta'$, contradiction. □

Let $(E, \mathscr{F})$ be an interval greedoid without minors (A)–(D). With $\mathscr{F}$ and a given basic word $x\alpha \in \mathscr{F}$ we associate an undirected rooted graph $G = (V, E)$ which we define inductively as follows: Let $G/x\alpha$ be the graph with node set $\{r\}$ and a loop for every element not in $x\alpha$. Suppose $G/x = (V', E\backslash x)$ has already been defined. Let $V := V' \cup \{r'\}$, and $x = (r, r')$ be a new edge. Consider an edge $e = (r, u)$ of G/x. If $e \notin \mathscr{F}$ and $u \neq r$, then replace (r, u) by (r', u). If $e \in \mathscr{F}$ and $u = r$, then replace (r, r) by (r, r'). Leaving the other edges unchanged, we obtain the graph $G = (V, E)$. Note that the contraction of the edge x in G yields G/x.

Proof of the Theorem. We claim that $(E, \mathscr{F})$ is the branching greedoid of G. In view of Lemma 8, it suffices to show that the paths and the parallel elements in $\mathscr{F}$ and G coincide.

We may assume that the assertion is true for $\mathscr{F}/x$ and G/x (see the construction above). For paths of length one and paths containing x, the claim is easily verified.

So let $k \geq 2$ and assume that the paths shorter than k coincide in $\mathscr{F}$ and G. Let $\alpha = x_1 \cdots x_k$ be a path in G. If α is a path in G/x, i.e. $\alpha \in \mathscr{P}/x$, then by Lemma 4 $\alpha \in \mathscr{P}$ or $x\alpha \in \mathscr{P}$. However, the latter case cannot occur since $x_1 \in \mathscr{F}$.

Therefore we can now assume that α is not a path in G/x. We thus have the

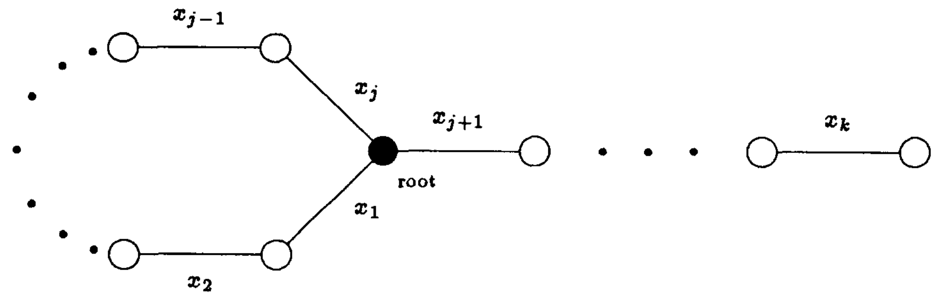

Fig. 1.

subgraph shown in *Fig. 1* in G/x, where $1 \leq j \leq k$. Since the branchings in G/x are the feasible sets in $\mathcal{F}/x$ we know that $\{x, x_1, \ldots, x_k\} \setminus \{x_j\} \in \mathcal{F}$. Augmenting $\{x_1, \ldots, x_{k-1}\}$ from the latter set gives $\{x, x_1, \ldots, x_{k-1}\} \in \mathcal{F}$ or $\{x_1, \ldots, x_k\} \in \mathcal{F}$. If $\{x_1, \ldots, x_k\} \in \mathcal{F}$, it contains an x_k-path. However, this path has length k, since strictly shorter paths coincide in $\mathcal{F}$ and G, thus $\alpha \in \mathcal{P}$.

If $\{x, x_1, \ldots, x_{k-1}\} \in \mathcal{F}$, then clearly $j = k$. Applying Lemma 5 to $x_1 \in \mathcal{F}$, $xx_kx_{k-1} \cdots x_2 \in \mathcal{P}$ (since $x_kx_{k-1} \cdots x_2 \in \mathcal{P}/x$) and $\{x, x_1, \ldots, x_k\} \notin \mathcal{F}$, yields $\alpha = x_1x_2 \cdots x_k \in \mathcal{P}$.

Conversely, let $\alpha = x_1 \cdots x_k \in \mathcal{P}$ with $k \geq 2$. If $\alpha \in \mathcal{P}/x$, then α or $x\alpha$ is a path in G (Lemma 4), but $x\alpha$ is not a path in G since x_1 is incident with the root. If $\alpha \notin \mathcal{P}/x$, by Lemma 4, $\alpha \notin \mathcal{F}/x$. Choose the index j such that $xx_1 \cdots x_{j-1} \in \mathcal{F}$, $xx_1 \cdots x_j \notin \mathcal{F}$. Lemma 5 implies that $x_jx_{j-1} \cdots x_2 \in \mathcal{P}$ and $\{x, x_1, \ldots, x_k\} \setminus x_j \in \mathcal{F}$, i.e. $\{x_1, \ldots, x_k\} \setminus x_j \in \mathcal{F}/x$. Hence $\{x_1, \ldots, x_k\} \setminus x_j$ is a branching in G/x and, by construction, $\{x, x_1, \ldots, x_k\} \setminus x_j$ is a branching in G. To see that α is a path in G, we can conclude as before, keeping in mind that the branching greedoid of G is an interval greedoid without minors (A)–(D).

It remains to show that $a \parallel b$ in $\mathcal{F}$ if and only if $a \parallel b$ in G. For $x = a$ or $x = b$, this is obvious from the construction. If $a \parallel b$ in $\mathcal{F}$ then, trivially, $a \parallel b$ in $\mathcal{F}/x$. By induction, $a \parallel b$ in G/x. If now a and b are not parallel in G then, by construction, exactly one of a, b is incident to the root. This, however, contradicts $a \parallel b$ in $\mathcal{F}$. Conversely, let $a \parallel b$ in G. Then $a \parallel b$ in G/x and hence $a \parallel b$ in $\mathcal{F}/x$. By Lemma 8, there exist αa, $\alpha b \in \mathcal{P}/x$, $\alpha ab \notin \mathcal{F}/x$. Thus $\alpha a \in \mathcal{P}$ or $x\alpha a \in \mathcal{P}$ (Lemma 5). If $\alpha a \in \mathcal{P}$ and $x\alpha b \in \mathcal{P}$ or $x\alpha a \in \mathcal{P}$ and $\alpha b \in \mathcal{P}$, a and b are not parallel in G, since the paths of G and $\mathcal{F}$ coincide. If αa, $\alpha b \in \mathcal{P}$ then $\alpha ab \notin \mathcal{F}$ since othewise $\{a, b, x\}$ induce minor (D) in $\mathcal{F}/\alpha$. Hence, by Lemma 8, $a \parallel b$ in $\mathcal{F}$. If $x\alpha a$, $x\alpha b \in \mathcal{P}$ then $a \parallel b$ in $\mathcal{F}$ since $x\alpha ab \notin \mathcal{F}$.

Thus, the requirements of Lemma 8 are fulfilled and $\mathcal{F}$ is the branching greedoid of G. □

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